Financial Time Series Models

Jorge Caiado
CEMAPRE/ISEG, University of Lisbon
Email: jcaiado@iseg.ulisboa.pt
Web: http://pascal.iseg.ulisboa.pt/~jcaiado/

Outline

- Introduction do financial time series analysis
- ARCH model
- GARCH model
- GARCH-M model
- Integrated GARCH model
- Exponential GARCH model
- Threshold GARCH model
- Power GARCH
- Component-GARCH
- Financial Applications

Introduction. ARCH Model

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

Characteristics of volatility

- Not directly observable
- *'Volatility clusters'*
- Stationary (within some fixed range)
- *'Leverage effect'*

Conditional mean and variance

$$r_t | r_{t-1}, r_{t-2}, \ldots \sim N(\mu, \sigma_t^2)$$

$$\mu = E(r_t | F_{t-1})$$

$$\sigma_t^2 = Var(r_t | F_{t-1}) = E[(r_t - \mu)^2 | F_{t-1}]$$

where F_{t-1} consists of all linear functions of the past returns.

Autoregressive conditional heteroscedasticity ARCH(q) Model, Engle (1982)

$$r_t = \mu + \sigma_t \epsilon_t$$

$u_t = \sigma_t \epsilon_t$ with $u_t = r_t - \mu$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i u_{t-i}^2,$$

where ϵ_t is a Gaussian white noise with zero mean and unit variance; $\alpha_0 > 0$; $\alpha_i \geq 0$.

ARCH Model

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

Properties of ARCH(1) model

$E(u_t) = 0$

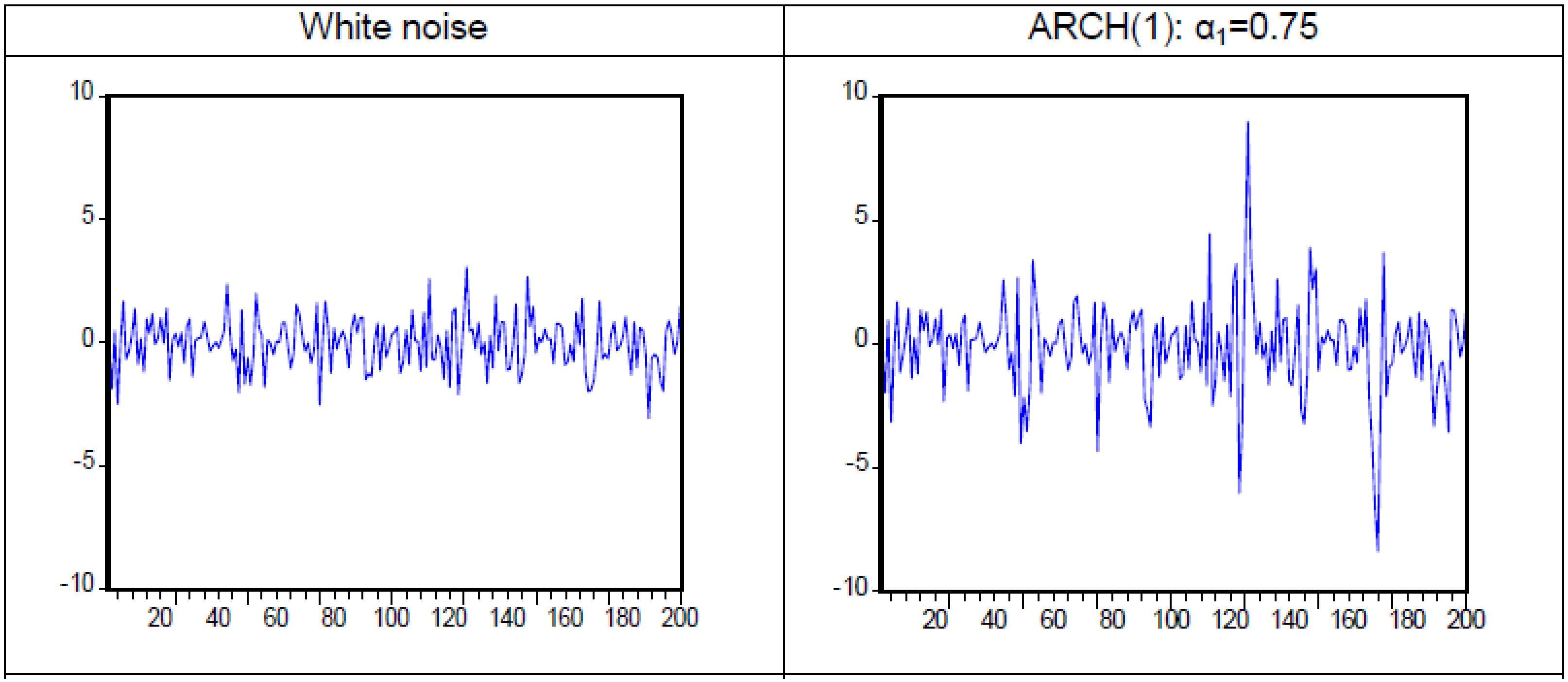
$$Var(u_t) = E(u_t^2) = \alpha_0 + \alpha_1 E(u_{t-1}^2) \Rightarrow Var(u_t) = \frac{\alpha_0}{1-\alpha_1}$$

$$E(u_t^4) = \frac{3\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)}$$

Identification of the ARCH(*q*) model

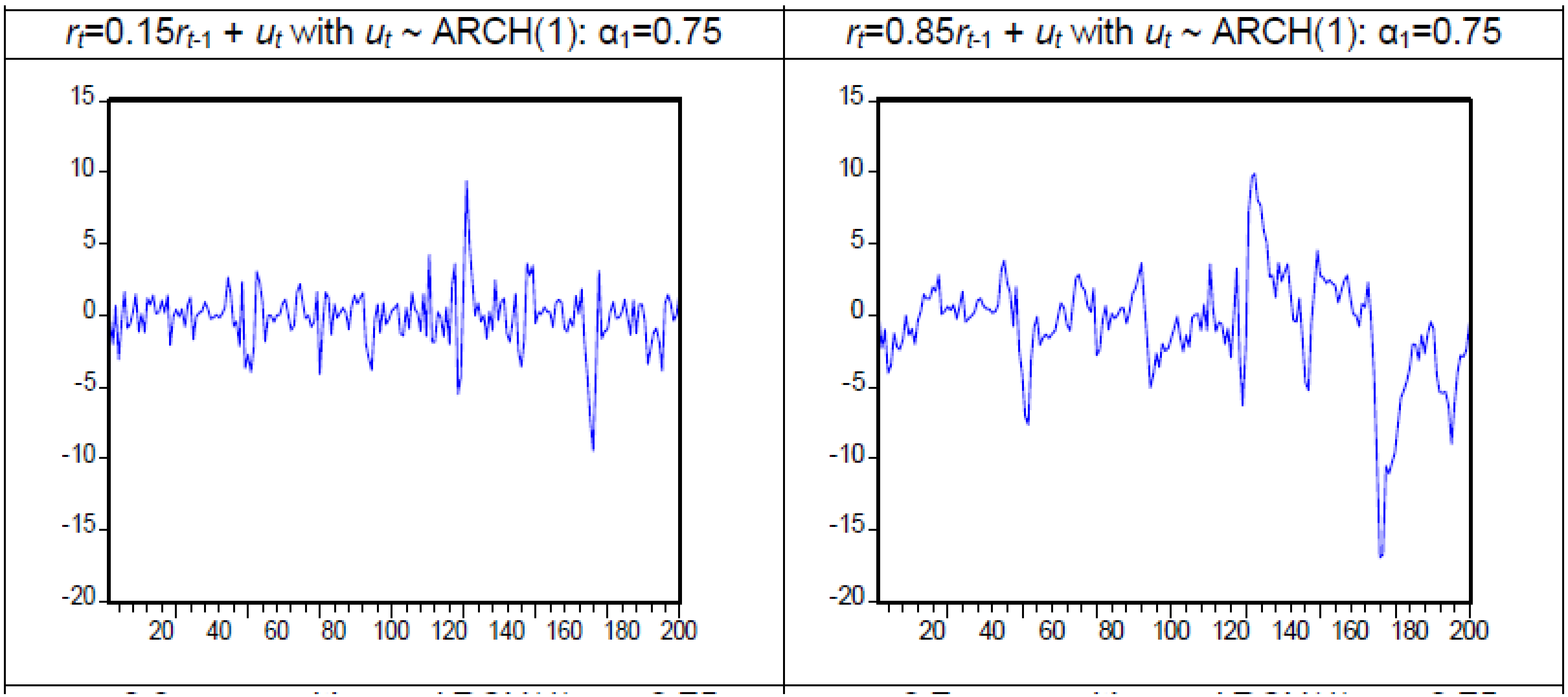
1) Estimate the model $r_t = \mu_t + u_t$, assuming σ_t^2 constant (if necessary, building an ARMA model for the return series)
2) Compute the squared residuals, $\hat{u}_t^2 = (r_t - \hat{\mu}_t)^2$
3) Calculate the ACF and PACF of $\hat{u}_t^2$ and identify the order *q*

ARCH Model


White noise
ARCH(1): α1=0.75
10
5
0
-5
-10
20
40
60
80
100
120
140
160
180
200

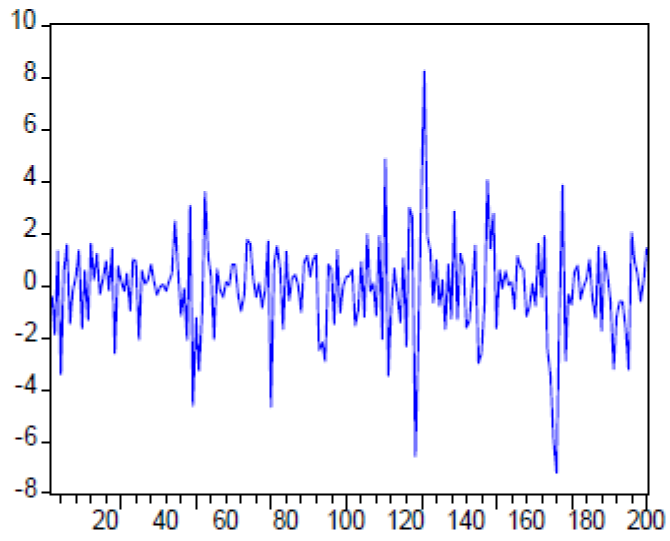
ARCH Model

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT
ISEG SINCE 1911

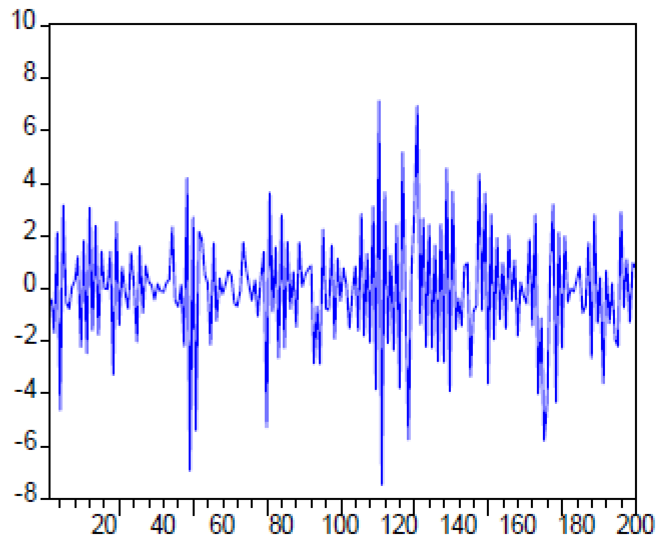


ARCH Model

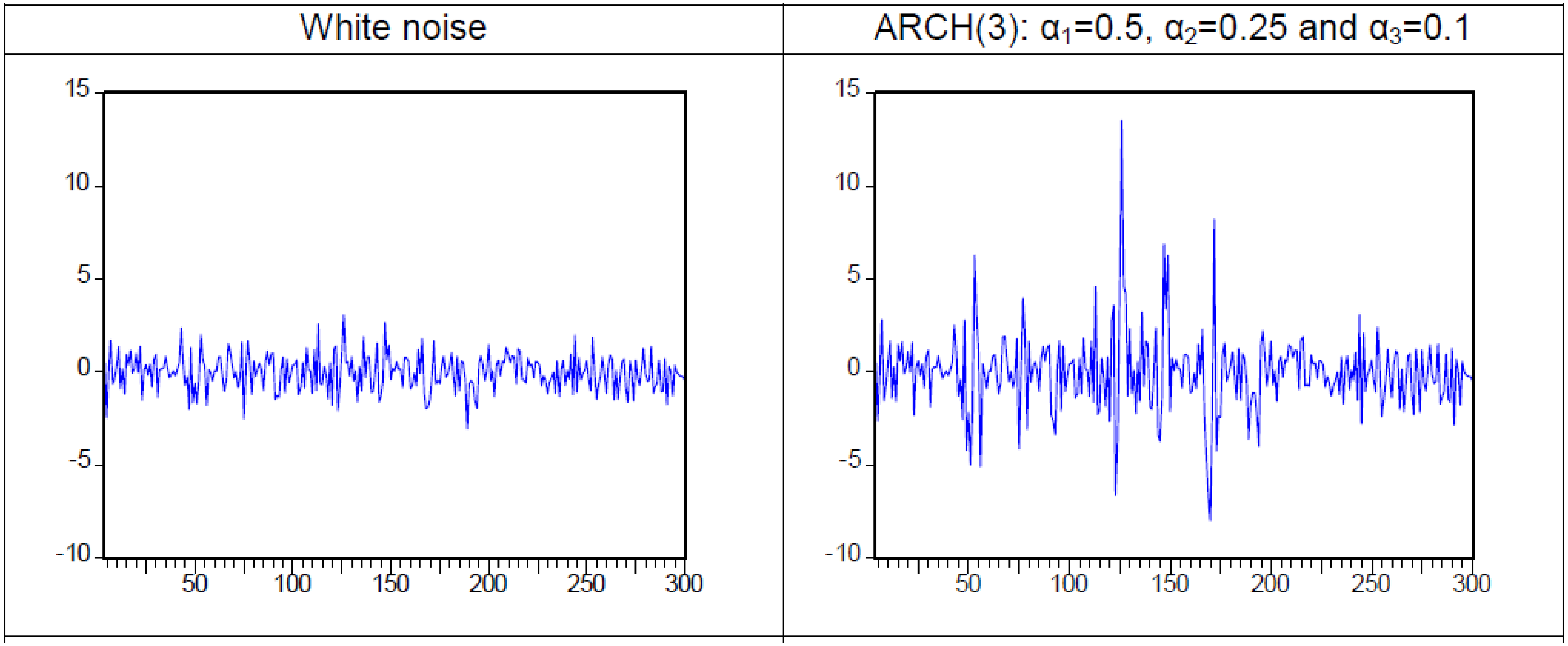
r_t=-0.2r_{t-1} + u_t with u_t ~ ARCH(1): α_1=0.75



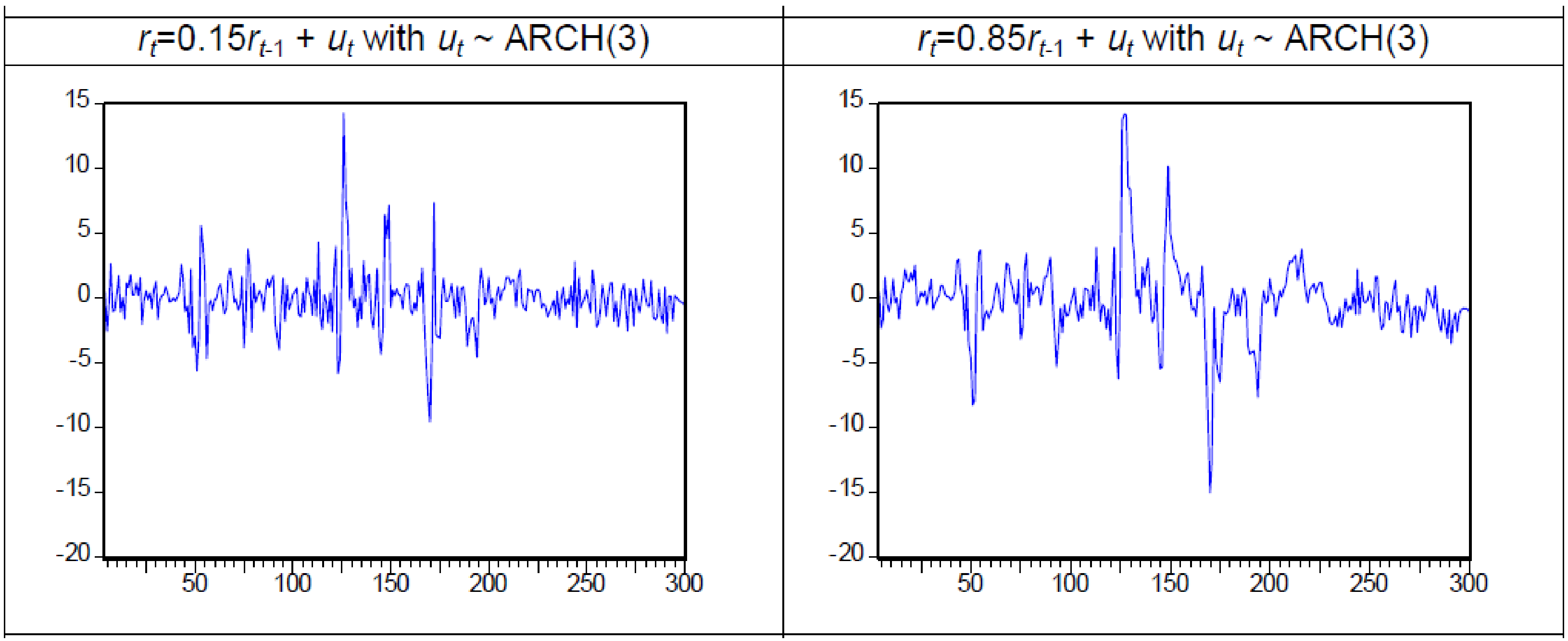
r_t=-0.7r_{t-1} + u_t with u_t ~ ARCH(1): α_1=0.75



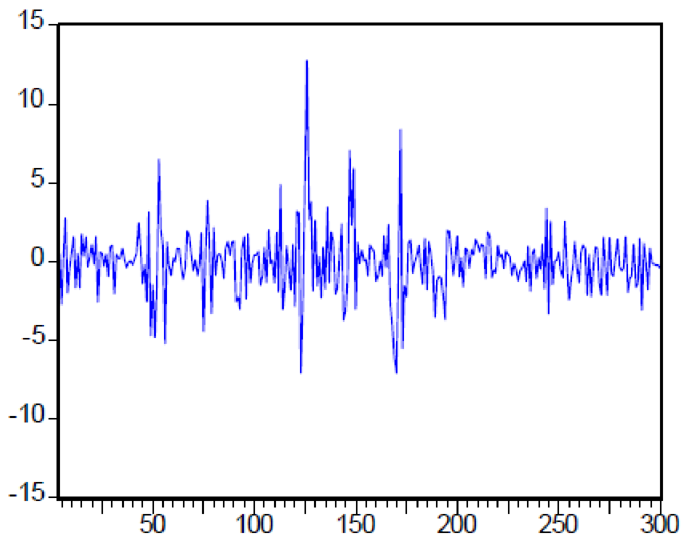
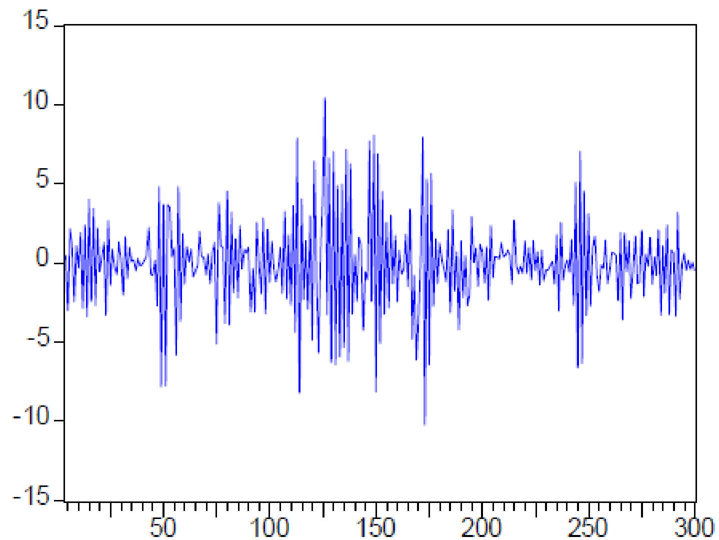
ARCH Model


White noise
ARCH(3): α1=0.5, α2=0.25 and α3=0.1
15
10
5
0
-5
-10
50
100
150
200
250
300

ARCH Model


r_t=0.15r_{t-1} + u_t with u_t ~ ARCH(3)
r_t=0.85r_{t-1} + u_t with u_t ~ ARCH(3)
15
10
5
0
-5
-10
-15
-20
50
100
150
200
250
300

ARCH Model

r_t=-0.2r_{t-1} + u_t with u_t ~ ARCH(3)	r_t=-0.7r_{t-1} + u_t with u_t ~ ARCH(3)
	

ARCH Model

Estimation of ARCH model (conditional maximum likelihood estimation, see Tsay, 2010)

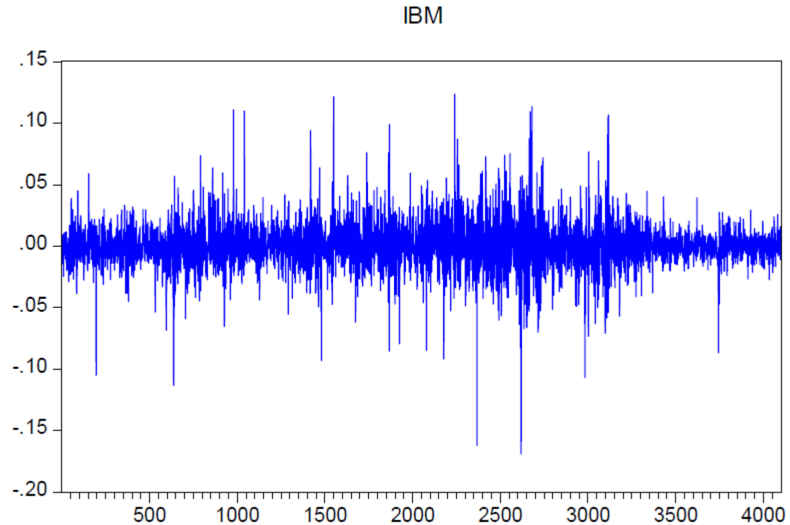
Model checking

- Ljung-Box (LB) test for standardized residuals (check the adequacy of the mean equation)
- Ljung-Box (LB) test for standardized squared residuals (check the adequacy of the conditional variance or volatility equation)
- Skewness, kurtosis and QQ-plot (check the validity of the distribution assumption)

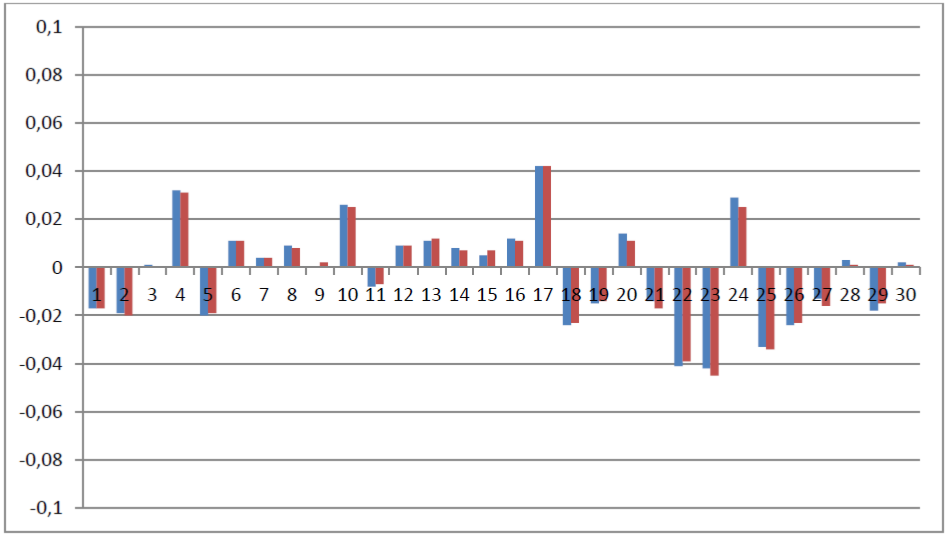
Forecasting

$\sigma_m^2(h) = \alpha_0 + \sum_{i=1}^{q} \alpha_i \sigma_m^2(h-i)$, where $\sigma_m^2(h-i) = u_{m+h-i}^2$ with $h - i \leq 0$

ARCH Model

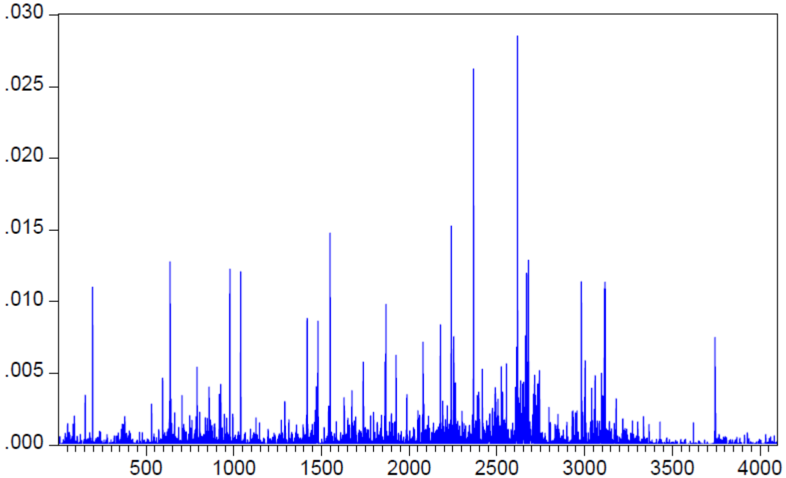


Returns of IBM from 11/6/1990 to 12/9/2006 (4099 daily observations)

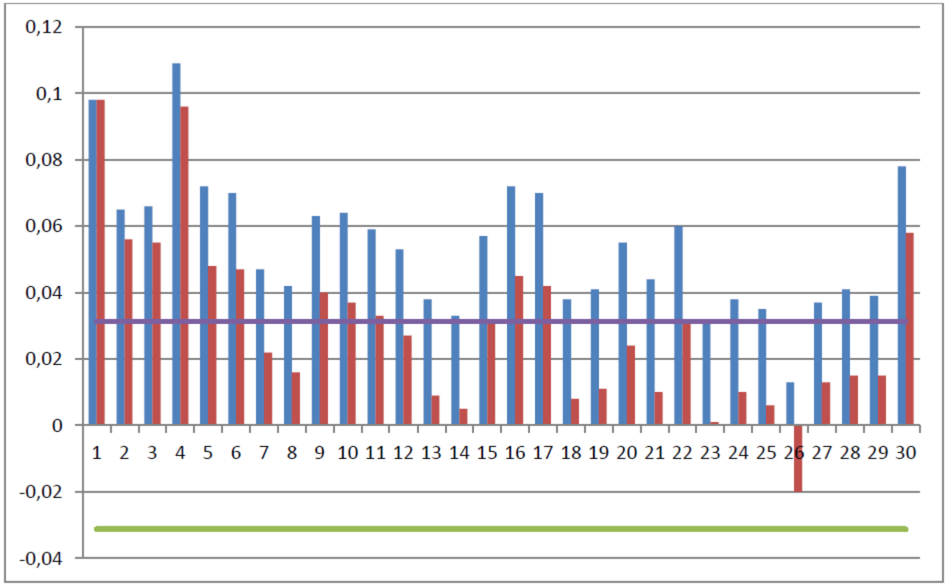


Sample ACF (blue bars) and sample PACF (red bars) of IBM returns

ARCH Model



Squared returns of IBM from 11/6/1990 to 12/9/2006 (4099 daily observations)



Sample ACF (blue bars) and sample PACF (red bars) of IBM squared returns and $\pm 2/n^{0.5}$ limits

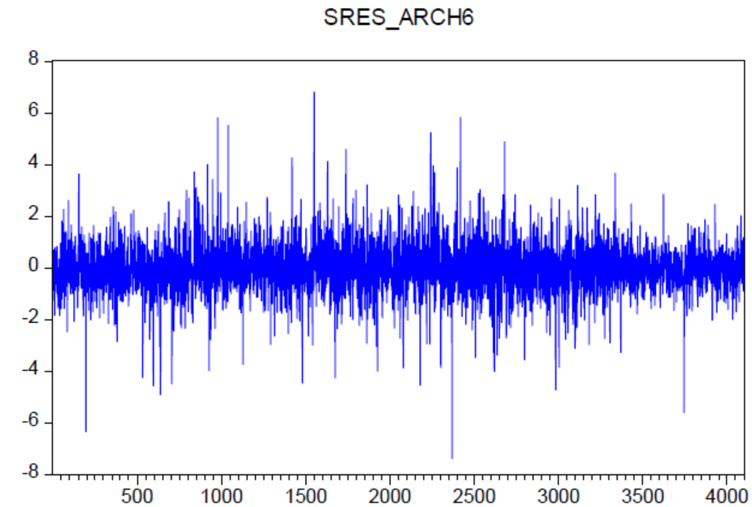
ARCH Model

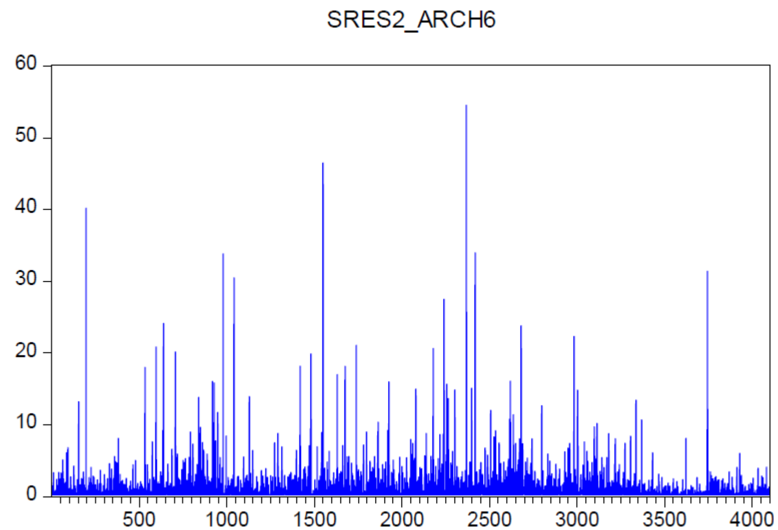
Dependent Variable: DLOG(IBM)
Method: ML - ARCH (Marquardt) - Normal distribution
Sample (adjusted): 2 4100
Included observations: 4099 after adjustments
Convergence achieved after 19 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*RESID(-2)^2 + C(5)*RESID(-3)^2 + C(6)*RESID(-4)^2 + C(7)*RESID(-5)^2 + C(8)*RESID(-6)^2

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000620	0.000230	2.699423	0.0069
Variance Equation				
C	0.000118	5.05E-06	23.44628	0.0000
RESID(-1)^2	0.221384	0.015811	14.00216	0.0000
RESID(-2)^2	0.119624	0.012652	9.454575	0.0000
RESID(-3)^2	0.053080	0.010671	4.974241	0.0000
RESID(-4)^2	0.177684	0.012546	14.16271	0.0000
RESID(-5)^2	0.100448	0.015700	6.398028	0.0000
RESID(-6)^2	0.147201	0.008053	18.27883	0.0000

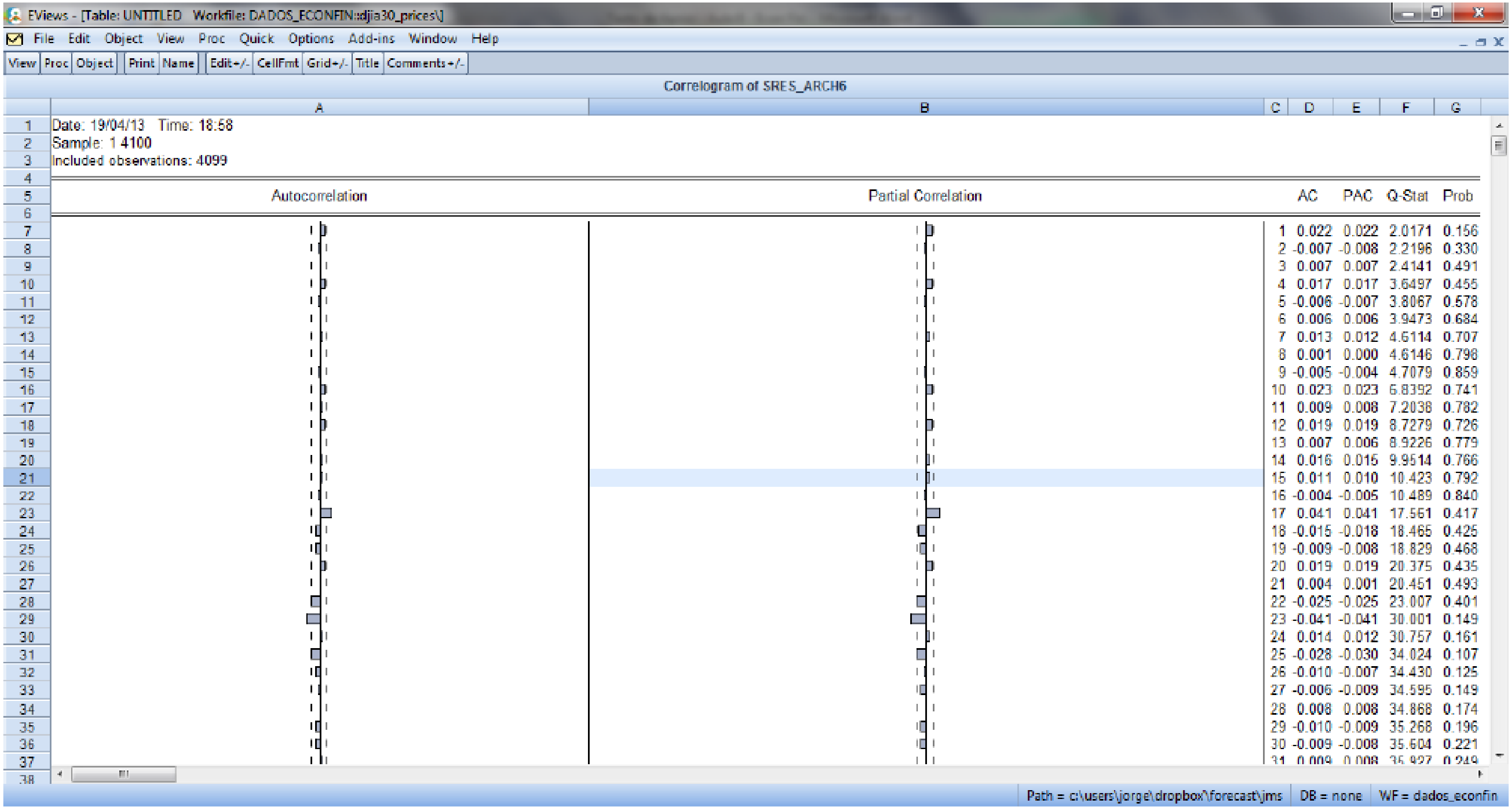
R-squared	-0.000249	Mean dependent var	0.000311
Adjusted R-squared	-0.000249	S.D. dependent var	0.019607
S.E. of regression	0.019609	Akaike info criterion	-5.177547
Sum squared resid	1.575812	Schwarz criterion	-5.165215
Log likelihood	10619.38	Hannan-Quinn criter.	-5.173181
Durbin-Watson stat	2.033389		

Estimates of ARCH(6) model for IBM log returns



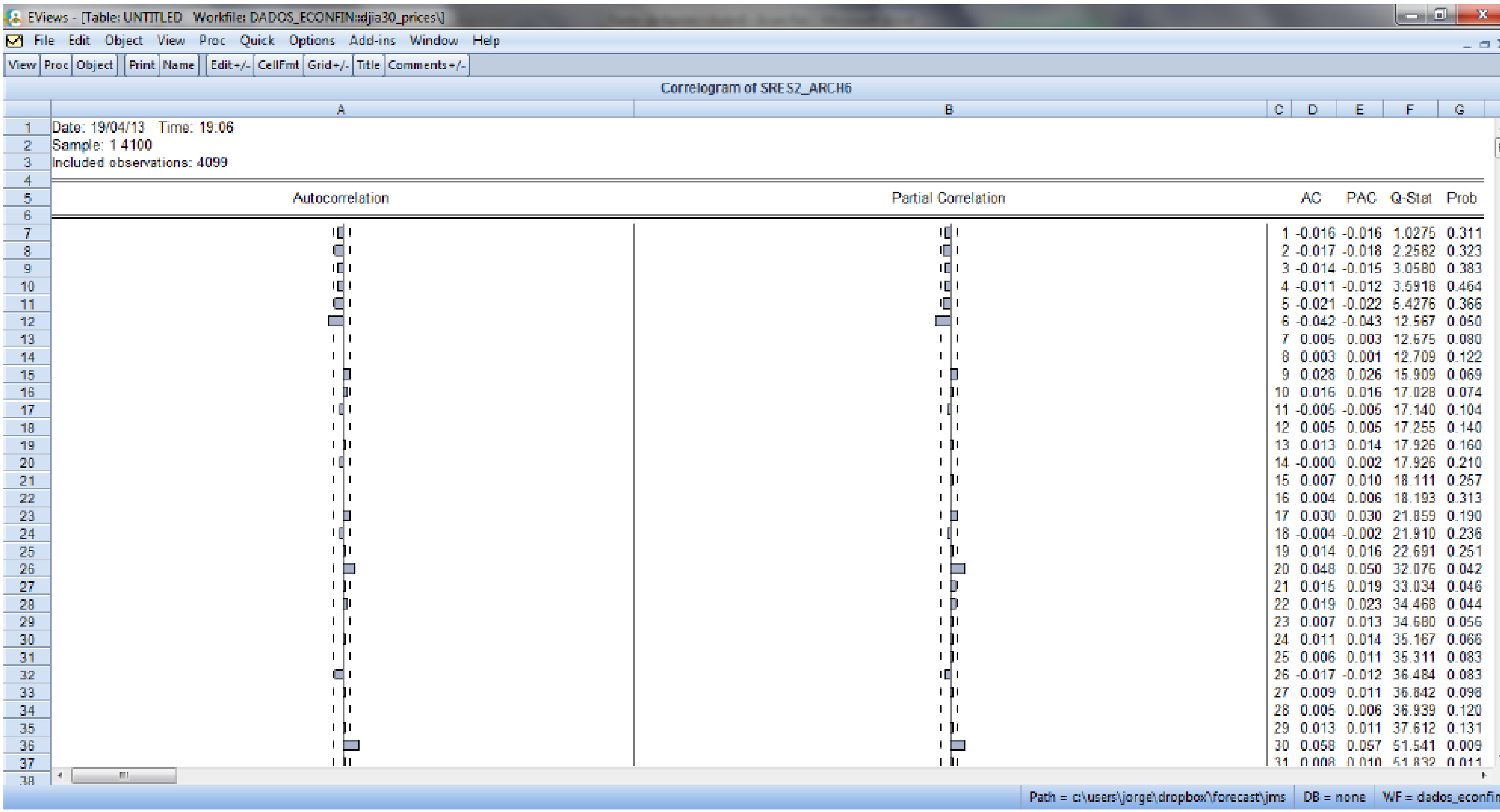


ARCH Model



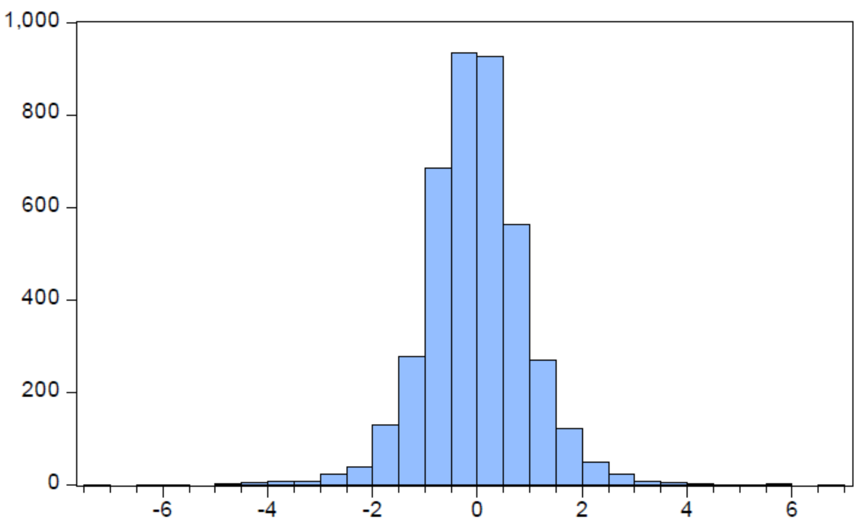
	AC	PAC	Q-Stat	Prob
1	0.022	0.022	2.0171	0.156
2	-0.007	-0.008	2.2196	0.330
3	0.007	0.007	2.4141	0.491
4	0.017	0.017	3.6497	0.455
5	-0.006	-0.007	3.8067	0.578
6	0.006	0.006	3.9473	0.684
7	0.013	0.012	4.6114	0.707
8	0.001	0.000	4.6146	0.798
9	-0.005	-0.004	4.7079	0.859
10	0.023	0.023	6.8392	0.741
11	0.009	0.008	7.2038	0.782
12	0.019	0.019	8.7279	0.726
13	0.007	0.006	8.9226	0.779
14	0.016	0.015	9.9514	0.766
15	0.011	0.010	10.423	0.792
16	-0.004	-0.005	10.489	0.840
17	0.041	0.041	17.561	0.417
18	-0.015	-0.018	18.465	0.425
19	-0.009	-0.008	18.829	0.468
20	0.019	0.019	20.375	0.435
21	0.004	0.001	20.451	0.493
22	-0.025	-0.025	23.007	0.401
23	-0.041	-0.041	30.001	0.149
24	0.014	0.012	30.757	0.161
25	-0.028	-0.030	34.024	0.107
26	-0.010	-0.007	34.430	0.125
27	-0.006	-0.009	34.595	0.149
28	0.008	0.008	34.868	0.174
29	-0.010	-0.009	35.268	0.196
30	-0.009	-0.008	35.604	0.221
31	0.009	0.008	35.927	0.249

ARCH Model



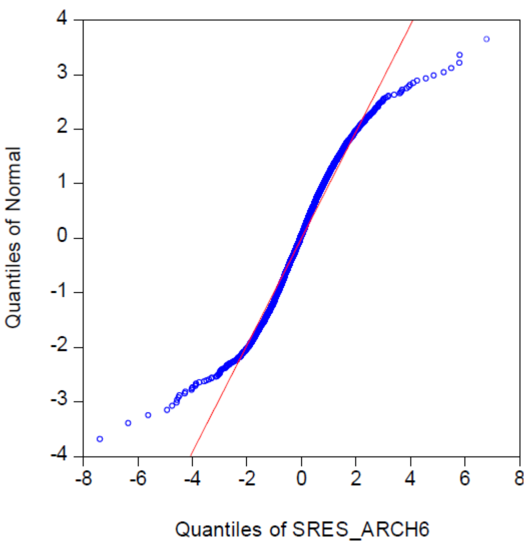
	AC	PAC	Q-Stat	Prob
1	-0.016	-0.016	1.0275	0.311
2	-0.017	-0.018	2.2562	0.323
3	-0.014	-0.015	3.0580	0.383
4	-0.011	-0.012	3.5918	0.464
5	-0.021	-0.022	5.4276	0.366
6	-0.042	-0.043	12.567	0.050
7	0.005	0.003	12.675	0.080
8	0.003	0.001	12.709	0.122
9	0.028	0.026	15.909	0.069
10	0.016	0.016	17.028	0.074
11	-0.005	-0.005	17.140	0.104
12	0.005	0.005	17.255	0.140
13	0.013	0.014	17.926	0.160
14	-0.000	0.002	17.926	0.210
15	0.007	0.010	18.111	0.257
16	0.004	0.006	18.193	0.313
17	0.030	0.030	21.859	0.190
18	-0.004	-0.002	21.910	0.236
19	0.014	0.016	22.691	0.251
20	0.048	0.050	32.076	0.042
21	0.015	0.019	33.034	0.046
22	0.019	0.023	34.468	0.044
23	0.007	0.013	34.680	0.056
24	0.011	0.014	35.167	0.066
25	0.006	0.011	35.311	0.083
26	-0.017	-0.012	36.484	0.083
27	0.009	0.011	36.842	0.096
28	0.005	0.006	36.939	0.120
29	0.013	0.011	37.612	0.131
30	0.058	0.057	51.541	0.009
31	0.008	0.010	51.832	0.011

ARCH Model



Series: SRES_ARCH6
Sample 1 4100
Observations 4099

Mean	-0.025992
Median	-0.034364
Maximum	6.816197
Minimum	-7.383901
Std. Dev.	0.999788
Skewness	0.016838
Kurtosis	7.439637
Jarque-Bera	3366.562
Probability	0.000000



GARCH Model

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

GARCH(*p*,*q*) model, Bollerselev (1986)

$r_t = \mu + \sigma_t \epsilon_t$

$u_t = \sigma_t \epsilon_t$ with $u_t = r_t - \mu$

$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i u_{t-i}^2 + \sum_{j=1}^{p} \beta_i \sigma_{t-j}^2,$

where ϵ_t is a Gaussian white noise with zero mean and unit variance; $\alpha_0 > 0$; $\alpha_i \geq 0$ and $\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1$.

In most applications, the simple GARCH (1,1) model has been found to provide a good representation of a wide variety of volatility processes even over very long periods as discussed in Bollerslev, Chou and Kroner (1992):

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

GARCH Model

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

Properties of a GARCH(1,1) model:

- The process is stationary if $\alpha_1 + \beta_1 < 1$;
- The unconditional variance is finite;
- The correlation between r_t and r_{t+k} is zero for all $k>0$;
- The correlation between $u_t^2 = (r_t - \mu)^2$ and u_{t+k}^2 is positive for all $k>0$;
- The unconditional kurtosis, $E(u_t^4)/[E(u_t^2)]^2$, exceed 3.

Forecasting volatility with a GARCH(1,1) model

$$\sigma_m^2(1) = \alpha_0 + \alpha_1 u_m^2 + \beta_1 \sigma_m^2$$

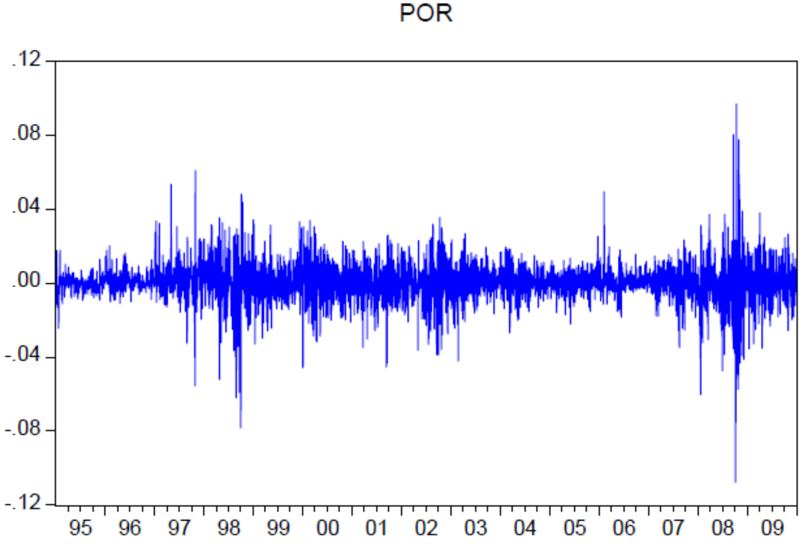
$$\sigma_m^2(2) = \alpha_0 + (\alpha_1 + \beta_1)\sigma_m^2(1)$$

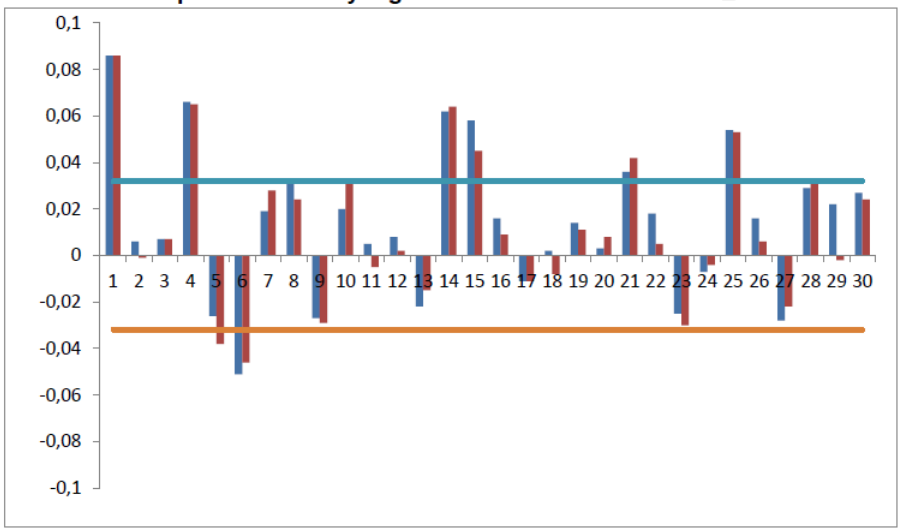
$$\sigma_m^2(h) = \alpha_0 + (\alpha_1 + \beta_1)\sigma_m^2(h-1), \quad h > 1$$

The stationary GARCH(1,1) has an AR(∞) representation.

GARCH Model

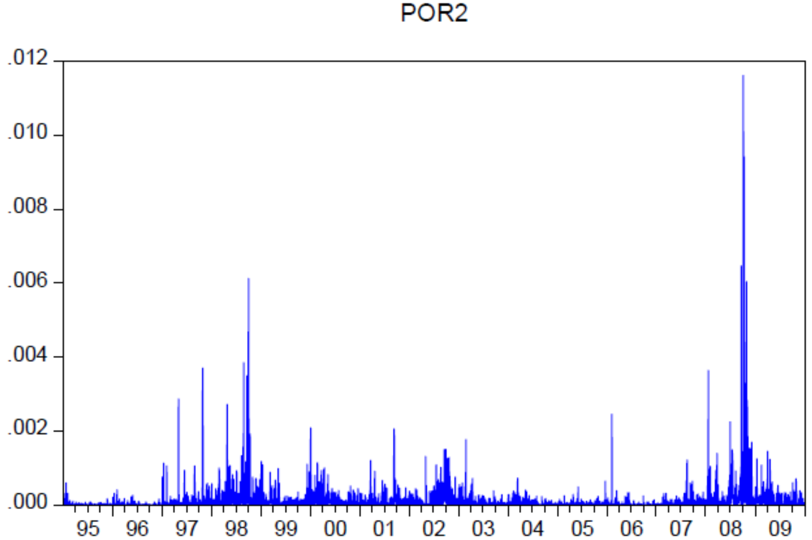
Daily log returns of PSI20 index (3/1/1995-31/12/2009, 3913 obs.)



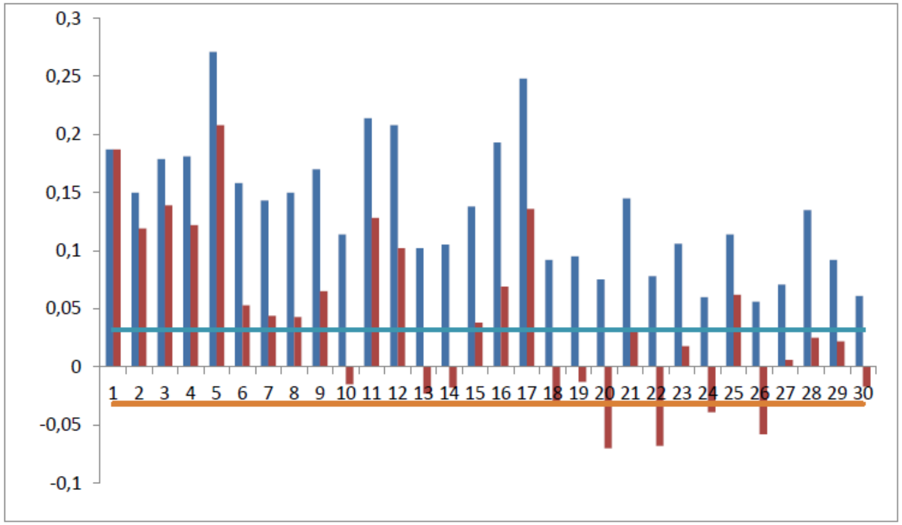


GARCH Model

Squared log returns of PSI20 index (3/1/1995-31/12/2009, 3913 obs.)



Sample ACF of squared log returns of PSI20 index and ±2sd



GARCH Model

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

ISEG SINCE 1911

AR(4)-GARCH(1,1) model with standard normal innovations ϵ_t

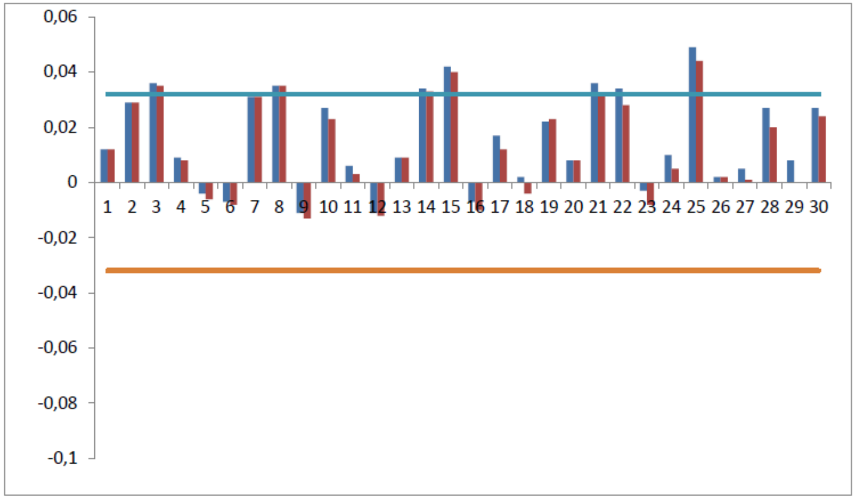
Dependent Variable: DLOG(POR)
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 27/04/13 Time: 22:33
Sample (adjusted): 9/01/1995 31/12/2009
Included observations: 3909 after adjustments
Convergence achieved after 14 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(4) + C(5)*RESID(-1)^2 + C(6)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000534	0.000144	3.706282	0.0002
AR(1)	0.100503	0.015171	6.624750	0.0000
AR(4)	0.049193	0.016714	2.943265	0.0032
Variance Equation				
C	6.27E-07	1.23E-07	5.085357	0.0000
RESID(-1)^2	0.094974	0.005449	17.42925	0.0000
GARCH(-1)	0.905888	0.005218	173.6213	0.0000

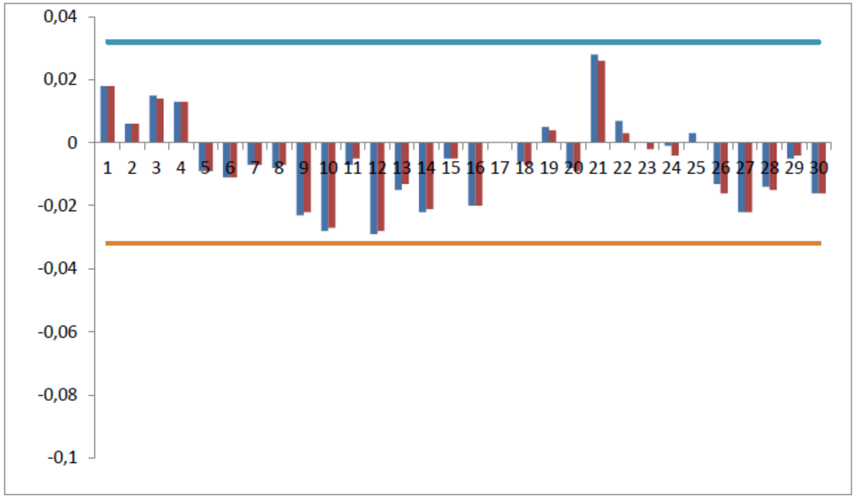
R-squared	0.010303	Mean dependent var	0.000159
Adjusted R-squared	0.009797	S.D. dependent var	0.011003
S.E. of regression	0.010949	Akaike info criterion	-6.578578
Sum squared resid	0.468292	Schwarz criterion	-6.568952
Log likelihood	12863.83	Hannan-Quinn criter.	-6.575162
Durbin-Watson stat	2.023774		

Inverted AR Roots	.50	.02-.47i	.02+.47i	-.45

Sample ACF of standardized residuals and ±2sd



Sample ACF of the squared standardized residuals and ±2sd



GARCH Model

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT
ISEG SINCE 1911

AR(4)-GARCH(1,1) model with standardized *t*-distribution innovations ϵ_t

Dependent Variable: DLOG(POR)
Method: ML - ARCH (Marquardt) - Student's t distribution
Date: 27/04/13 Time: 22:53
Sample (adjusted): 9/01/1995 31/12/2009
Included observations: 3909 after adjustments
Convergence achieved after 14 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(4) + C(5)*RESID(-1)^2 + C(6)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000462	0.000124	3.718221	0.0002
AR(1)	0.078666	0.015689	5.014209	0.0000
AR(4)	0.034255	0.015969	2.145130	0.0319
Variance Equation				
C	4.32E-07	1.43E-07	3.026173	0.0025
RESID(-1)^2	0.086051	0.008961	9.603285	0.0000
GARCH(-1)	0.916242	0.007797	117.5049	0.0000
T-DIST. DOF	5.875113	0.512806	11.45679	0.0000

R-squared	0.010018	Mean dependent var	0.000159
Adjusted R-squared	0.009511	S.D. dependent var	0.011003
S.E. of regression	0.010951	Akaike info criterion	-6.642025
Sum squared resid	0.468427	Schwarz criterion	-6.630795
Log likelihood	12988.84	Hannan-Quinn criter.	-6.638039
Durbin-Watson stat	1.981680		

Inverted AR Roots	.45	.02-.43i	.02+.43i	-.41

GARCH Model

AR(4)-GARCH(1,1) model with generalized error distribution (GED) innovations ϵ_t

Dependent Variable: DLOG(POR)
Method: ML - ARCH (Marquardt) - Generalized error distribution (GED)
Date: 27/04/13 Time: 22:55
Sample (adjusted): 9/01/1995 31/12/2009
Included observations: 3909 after adjustments
Convergence achieved after 14 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(4) + C(5)*RESID(-1)^2 + C(6)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000340	0.000117	2.912876	0.0036
AR(1)	0.069867	0.015018	4.652110	0.0000
AR(4)	0.030013	0.015249	1.968236	0.0490
Variance Equation				
C	4.98E-07	1.66E-07	2.997532	0.0027
RESID(-1)^2	0.089299	0.009191	9.716434	0.0000
GARCH(-1)	0.911957	0.008348	109.2453	0.0000
GED PARAMETER	1.264832	0.029680	42.61552	0.0000

R-squared	0.009915	Mean dependent var	0.000159
Adjusted R-squared	0.009408	S.D. dependent var	0.011003
S.E. of regression	0.010952	Akaike info criterion	-6.640488
Sum squared resid	0.468476	Schwarz criterion	-6.629258
Log likelihood	12985.83	Hannan-Quinn criter.	-6.636502
Durbin-Watson stat	1.965091		

Inverted AR Roots	.43	.02-.42i	.02+.42i	-.40

GARCH-M Model

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

GARCH(*p*,*q*)-M or GARCH *in the mean* model, Engle, Lilien and Robins, 1987

$r_t = \mu + c\sigma_t^2 + u_t,$

$u_t = \sigma_t \epsilon_t,$

$\sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + (1-\beta_1) u_{t-1}^2$

where c is the *risk premium* parameter. A positive c indicates that expected returns increase as risk (or volatility) increases.

Other specifications: $r_t = \mu + c\sigma_t + u_t$ and $r_t = \mu + c\ln(\sigma_t^2) + u_t$

GARCH-M Model

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

AR(4)-GARCH(1,1)-M model with σ_t in the mean equation

Dependent Variable: DLOG(POR)
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 02/05/13 Time: 17:29
Sample (adjusted): 9/01/1995 31/12/2009
Included observations: 3909 after adjustments
Convergence achieved after 26 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(5) + C(6)*RESID(-1)^2 + C(7)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
@SQRT(GARCH)	-0.009009	0.043632	-0.206486	0.8364
C	0.000591	0.000313	1.891050	0.0586
AR(1)	0.100434	0.015401	6.521343	0.0000
AR(4)	0.049062	0.016891	2.904585	0.0037
Variance Equation				
C	6.26E-07	1.23E-07	5.082977	0.0000
RESID(-1)^2	0.094848	0.005480	17.30703	0.0000
GARCH(-1)	0.906011	0.005235	173.0809	0.0000

R-squared	0.010526	Mean dependent var	0.000159
Adjusted R-squared	0.009765	S.D. dependent var	0.011003
S.E. of regression	0.010950	Akaike info criterion	-6.578080
Sum squared resid	0.468187	Schwarz criterion	-6.566850
Log likelihood	12863.86	Hannan-Quinn criter.	-6.574095
Durbin-Watson stat	2.024305		

Inverted AR Roots	.50	.02-.47i	.02+.47i	-.45

GARCH-M Model

AR(4)-GARCH(1,1)-M model with σ_t^2 in the mean equation

Dependent Variable: DLOG(POR)
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 02/05/13 Time: 17:33
Sample (adjusted): 9/01/1995 31/12/2009
Included observations: 3909 after adjustments
Convergence achieved after 31 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(5) + C(6)*RESID(-1)^2 + C(7)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
GARCH	0.461751	2.092436	0.220676	0.8253
C	0.000510	0.000176	2.902836	0.0037
AR(1)	0.100520	0.015450	6.506119	0.0000
AR(4)	0.049328	0.016928	2.913940	0.0036
Variance Equation				
C	6.29E-07	1.23E-07	5.096903	0.0000
RESID(-1)^2	0.095165	0.005490	17.33315	0.0000
GARCH(-1)	0.905696	0.005241	172.8241	0.0000

R-squared	0.009990	Mean dependent var	0.000159
Adjusted R-squared	0.009229	S.D. dependent var	0.011003
S.E. of regression	0.010953	Akaike info criterion	-6.578083
Sum squared resid	0.468441	Schwarz criterion	-6.566853
Log likelihood	12863.86	Hannan-Quinn criter.	-6.574098
Durbin-Watson stat	2.022740		

Inverted AR Roots	.50	.02-.47i	.02+.47i	-.45

AR(4)-GARCH(1,1)-M model with $\log(\sigma_t^2)$ in the mean equation

Dependent Variable: DLOG(POR)
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 02/05/13 Time: 17:34
Sample (adjusted): 9/01/1995 31/12/2009
Included observations: 3909 after adjustments
Convergence achieved after 21 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(5) + C(6)*RESID(-1)^2 + C(7)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
LOG(GARCH)	-9.19E-05	0.000170	-0.540617	0.5888
C	-0.000410	0.001748	-0.234297	0.8148
AR(1)	0.100335	0.015307	6.554609	0.0000
AR(4)	0.048923	0.016835	2.905968	0.0037
Variance Equation				
C	6.26E-07	1.23E-07	5.079524	0.0000
RESID(-1)^2	0.094820	0.005487	17.27976	0.0000
GARCH(-1)	0.906023	0.005245	172.7538	0.0000

R-squared	0.010829	Mean dependent var	0.000159
Adjusted R-squared	0.010069	S.D. dependent var	0.011003
S.E. of regression	0.010948	Akaike info criterion	-6.578155
Sum squared resid	0.468044	Schwarz criterion	-6.566925
Log likelihood	12864.00	Hannan-Quinn criter.	-6.574170
Durbin-Watson stat	2.024738		

Inverted AR Roots	.50	.02-.47i	.02+.47i	-.45

IGARCH Model

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

IGARCH(1,1) model, Nelson (1990)

$u_t = \sigma_t \epsilon_t$ with $u_t = r_t - \mu$

$\sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + (1 - \beta_1) u_{t-1}^2$

where ϵ_t is a Gaussian white noise with zero mean and unit variance;

$\alpha_0 > 0$; $\alpha_1 + \beta_1 = 1$ (or $\alpha_1 = 1 - \beta_1$); $0 < \beta_1 < 1$.

Mathematical properties of an IGARCH(1,1) model:

$u_t^2 \sim ARIMA(0,1,1) => u_t$ is not covariance stationary
However, the process is strictly stationary when E[log($\beta_1 + \alpha_1 \epsilon_t^2$)]

IGARCH Model

AR(4)-IGARCH(1,1) model with normal distribution

Dependent Variable: DLOG(POR)
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 02/05/13 Time: 17:28
Sample (adjusted): 9/01/1995 31/12/2009
Included observations: 3909 after adjustments
Convergence achieved after 29 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(4)*RESID(-1)^2 + (1 - C(4))*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000516	0.000117	4.397674	0.0000
AR(1)	0.102265	0.012642	8.089574	0.0000
AR(4)	0.051306	0.014576	3.519996	0.0004
		Variance Equation		
RESID(-1)^2	0.063926	0.002487	25.70252	0.0000
GARCH(-1)	0.936074	0.002487	376.3674	0.0000

R-squared	0.010395	Mean dependent var	0.000159
Adjusted R-squared	0.009889	S.D. dependent var	0.011003
S.E. of regression	0.010949	Akaike info criterion	-6.564866
Sum squared resid	0.468249	Schwarz criterion	-6.558449
Log likelihood	12835.03	Hannan-Quinn criter.	-6.562589
Durbin-Watson stat	2.027306		

Inverted AR Roots	.50	.03-.47i	.03+.47i	-.45

Exponential GARCH Model

Exponential GARCH or EGARCH(p,q) model. Nelson (1991) uses the ARMA(p,q) process to describe the conditional variance of returns,

$$\ln(\sigma_t^2) = \alpha_0 + \frac{1+\beta_1 B + \cdots + \beta_{q-1} B^{q-1}}{1-\alpha_1 B - \cdots - \alpha_p B^p} g(\varepsilon_{t-1})$$

where

$$g(\varepsilon_t) = \theta\varepsilon_t + \gamma\left[|\varepsilon_t| - E(|\varepsilon_t|)\right] = \begin{cases} (\theta+\gamma)\varepsilon_t - \gamma E(|\varepsilon_t|) & \text{if } \varepsilon_t \geq 0 \\ (\theta-\gamma)\varepsilon_t - \gamma E(|\varepsilon_t|) & \text{if } \varepsilon_t < 0 \end{cases}$$

is an asymmetric function of the volatility residuals, and θ and γ are real constants. The function $g(\varepsilon_t)$ has zero mean, because $E(|\varepsilon_t|) = \sqrt{2/\pi}$ when ε_t has a standardized normal distribution. The required expectation for the standardized *t*-Student distribution is

$$E\left[|\varepsilon_t|\right] = \frac{2\sqrt{\upsilon-2}\,\Gamma\left[(\upsilon+1)/2\right]}{(\upsilon-1)\Gamma\left[\upsilon/2\right]\sqrt{\pi}}$$

Exponential GARCH Model

A simple EGARCH(1,1) model can be written as

$$(1-\alpha_1 B)\ln(\sigma_t^2) = (1-\alpha_1)\alpha_0 + g(\varepsilon_{t-1}) = \begin{cases} \alpha^* + (\gamma+\theta)\varepsilon_{t-1} & \text{if } \varepsilon_{t-1} \geq 0 \\ \alpha^* + (\gamma-\theta)(-\varepsilon_{t-1}) & \text{if } \varepsilon_{t-1} < 0 \end{cases}$$

where ε_t has a standardized normal distribution and $\alpha^* = (1-\alpha_1)\alpha_0 - \sqrt{2/\pi}\gamma$.

An alternative form for EGARCH(p,q) model is

$$\ln(\sigma_t^2) = \alpha_0 + \sum_{j=1}^{p}\beta_j \ln(\sigma_{t-j}^2) + \sum_{i=1}^{q}\alpha_i \frac{|u_{t-i}| + \gamma_i u_{t-i}}{\sigma_{t-i}} \quad \text{with } u_t = \sigma_t \varepsilon_t$$

The γ_i parameter is the "leverage effect" of u_{t-i} . In many real application, we expect γ_i to be negative.

EGARCH(1,1): $\ln(\sigma_t^2) = \alpha_0 + \beta_1 \ln(\sigma_{t-1}^2) + \alpha_1\left[\left|\frac{u_{t-1}}{\sigma_{t-1}}\right| + \gamma_1 \frac{u_{t-1}}{\sigma_{t-1}}\right]$

Exponential GARCH Model

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

ISEG SINCE 1911

EGARCH(1,1) model assuming that ε_t follows a standardized *t*-Student distribution

Dependent Variable: DLOG(MSFT)
Method: ML - ARCH (Marquardt) - Student's t distribution
Date: 05/10/13 Time: 19:25
Sample (adjusted): 2 4100
Included observations: 4099 after adjustments
Convergence achieved after 13 iterations
Presample variance: backcast (parameter = 0.7)
LOG(GARCH) = C(2) + C(3)*ABS(RESID(-1)/@SQRT(GARCH(-1))) + C(4)*RESID(-1)/@SQRT(GARCH(-1)) + C(5)*LOG(GARCH(-1))

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000379	0.000254	1.492346	0.1356
Variance Equation				
C(2)	-0.170958	0.026696	-6.403773	0.0000
C(3)	0.140998	0.014416	9.780561	0.0000
C(4)	-0.024350	0.009129	-2.667174	0.0076
C(5)	0.991782	0.002697	367.7117	0.0000
T-DIST. DOF	6.541075	0.509888	12.82846	0.0000

R-squared	-0.000397	Mean dependent var	0.000820
Adjusted R-squared	-0.000397	S.D. dependent var	0.022159
S.E. of regression	0.022163	Akaike info criterion	-5.039211
Sum squared resid	2.012917	Schwarz criterion	-5.029962
Log likelihood	10333.86	Hannan-Quinn criter.	-5.035937
Durbin-Watson stat	2.022076		

Thershold GARCH Model

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

ISEG SINCE 1911

Modelo TGARCH(p,q). The threshold GARCH or TGARCH model were introduced independently by Glosten, Jagannathan e Runkle (1993) and Zakoian (1994), which allows for asymmetric shocks to volatility. The conditional variance for the simple TARCH(1,1) model is defined by

$$\sigma_t^2 = \alpha_0 + \alpha u_{t-1}^2 + \gamma S_{t-1} u_{t-1}^2 + \beta \sigma_{t-1}^2$$

where $S_{t-1} = 1$ if $u_{t-1} < 0$ and 0 otherwise. In this model, volatility tends to rise with the *bad news* ($\varepsilon_{t-1} < 0$) and to fall with the *good news* ($\varepsilon_{t-1} > 0$). Good news has an impact of α while bad news has an impact of $\alpha + \gamma$. This model is concerned with the leverage effect sometimes observed in stock returns. If $\gamma > 0$ then the leverage effect exists. If $\gamma \neq 0$, the shock is asymmetric, and if $\gamma = 0$, the shock is symmetric. The persistence of shocks to volatility is given by $\alpha + \beta + \gamma/2$.

Thershold GARCH Model

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT
ISEG SINCE 1911

A TGARCH(p,q) model assumes the form

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} (\alpha_i + \gamma_i S_{t-i}) u_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$$

TGARCH(1,1) model assuming standardized *t*-Student distribution

Dependent Variable: DLOG(MSFT)
Method: ML - ARCH (Marquardt) - Student's t distribution
Date: 05/10/13 Time: 19:28
Sample (adjusted): 2 4100
Included observations: 4099 after adjustments
Convergence achieved after 18 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*RESID(-1)^2*(RESID(-1)<0)
+ C(5)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000395	0.000252	1.568787	0.1167
Variance Equation				
C	1.61E-06	6.69E-07	2.403195	0.0163
RESID(-1)^2	0.054374	0.009195	5.913108	0.0000
RESID(-1)^2*(RESID(-1)<0)	0.032485	0.012880	2.522157	0.0117
GARCH(-1)	0.931878	0.007020	132.7436	0.0000
T-DIST. DOF	6.293033	0.493761	12.74510	0.0000

R-squared	-0.000368	Mean dependent var	0.000820
Adjusted R-squared	-0.000368	S.D. dependent var	0.022159
S.E. of regression	0.022163	Akaike info criterion	-5.031865
Sum squared resid	2.012861	Schwarz criterion	-5.022616
Log likelihood	10318.81	Hannan-Quinn criter.	-5.028591
Durbin-Watson stat	2.022133		

Power GARCH Model

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT
SINCE 1911

Power GARCH (PARCH) model, Taylor (1986) and Schwert (1989) introduced the standard deviation GARCH model, which is generalized in Ding, Granger and Engle (1993) with the Power ARCH model.

$$\sigma_t^{\delta} = \alpha_0 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^{\delta} + \sum_{i=1}^{p} \alpha_i (|u_{t-i}| - \gamma_i u_{t-i})^{\delta}$$

where $\delta > 0$, $|\gamma_i| \leq 1$ for $i = 1,\ldots k$, $\gamma_i = 0$ for all $i > k$ and $k \leq p$. Here δ is the power parameter of the standard deviation and γ is the leverage parameter (added to capture asymmetry).

The conditional variance in the simple PGARCH(1,1) model is

$$\sigma_t^{\delta} = \alpha_0 + \beta \sigma_{t-1}^{\delta} + \alpha (|u_{t-1}| - \gamma u_{t-1})^{\delta}$$

If $\delta = 2$ and $\gamma = 0$ the PARCH is simply a standard GARCH(1,1) model. The presence of leverage effects can be tested by the hypothesis that $\gamma < 0$. If $\gamma \neq 0$ the news impact is asymmetric.

Power GARCH Model

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT
ISEG SINCE 1911

PGARCH(1,1) model assuming standardized *t*-Student distribution

Dependent Variable: DLOG(MSFT)
Method: ML - ARCH (Marquardt) - Student's t distribution
Date: 05/10/13 Time: 19:30
Sample (adjusted): 2 4100
Included observations: 4099 after adjustments
Convergence achieved after 34 iterations
Presample variance: backcast (parameter = 0.7)
@SQRT(GARCH)^C(6) = C(2) + C(3)*(ABS(RESID(-1)) - C(4)*RESID(-1))^C(6) + C(5)*@SQRT(GARCH(-1))^C(6)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000362	0.000252	1.436174	0.1510
Variance Equation				
C(2)	8.63E-05	7.56E-05	1.141372	0.2537
C(3)	0.078654	0.008237	9.548318	0.0000
C(4)	0.180563	0.066506	2.714986	0.0066
C(5)	0.935244	0.006969	134.1960	0.0000
C(6)	1.083784	0.182892	5.925825	0.0000
T-DIST. DOF	6.502492	0.507147	12.82171	0.0000

R-squared	-0.000428	Mean dependent var	0.000820
Adjusted R-squared	-0.000428	S.D. dependent var	0.022159
S.E. of regression	0.022163	Akaike info criterion	-5.038755
Sum squared resid	2.012980	Schwarz criterion	-5.027965
Log likelihood	10333.93	Hannan-Quinn criter.	-5.034935
Durbin-Watson stat	2.022013		

Component GARCH Model

The component GARCH (CGARCH) model. The Component GARCH model of Engle and Lee (1999) was designed to account for long-run volatility dependencies.

The conditional variance in the GARCH(1,1) model can be written as

$$\sigma_t^2 - \sigma^2 = \alpha(u_{t-1}^2 - \sigma^2) + \beta(\sigma_{t-1}^2 - \sigma^2)$$

where $\sigma^2 = \omega/(1-\alpha-\beta)$ refers to the unconditional variance (i.e., σ_t^2 shows mean reversion to σ^2). By contrast the CGARCH(1,1) model is obtained by relaxing the assumption of a constant σ^2 (i.e., σ_t^2 allows mean reversion to a varying level m_t)

$$\sigma_t^2 - m_t = \alpha(u_{t-1}^2 - m_{t-1}) + \beta(\sigma_{t-1}^2 - m_{t-1})$$
$$m_t = \omega + \rho(m_{t-1} - \omega) + \phi(u_{t-1}^2 - \sigma_{t-1}^2)$$

where m_t is the time varying long-run volatility. The first equation refers to the **transitory component**, $\sigma_t^2 - m_t$, which converges to zero with powers of $\alpha + \beta$. The second equation refers to the **long-run component**, m_t, which converges to ω with powers of ρ.

The component GARCH can be expressed as a restricted GARCH(2,2) model.

Component GARCH Model

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

CGARCH(1,1) model assuming standardized *t*-Student distribution

Dependent Variable: DLOG(MSFT)
Method: ML - ARCH (Marquardt) - Student's t distribution
Date: 05/10/13 Time: 19:32
Sample (adjusted): 2 4100
Included observations: 4099 after adjustments
Convergence achieved after 33 iterations
Presample variance: backcast (parameter = 0.7)
Q = C(2) + C(3)*(Q(-1) - C(2)) + C(4)*(RESID(-1)^2 - GARCH(-1))
GARCH = Q + C(5) * (RESID(-1)^2 - Q(-1)) + C(6)*(GARCH(-1) - Q(-1))

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000450	0.000244	1.844423	0.0651
Variance Equation				
C(2)	0.002407	0.024802	0.097044	0.9227
C(3)	0.999956	0.000494	2022.935	0.0000
C(4)	0.024688	0.005284	4.671983	0.0000
C(5)	0.067142	0.014005	4.794254	0.0000
C(6)	0.839036	0.039048	21.48715	0.0000
T-DIST. DOF	6.743333	0.462793	14.57095	0.0000

R-squared	-0.000279	Mean dependent var	0.000820
Adjusted R-squared	-0.000279	S.D. dependent var	0.022159
S.E. of regression	0.022162	Akaike info criterion	-5.037872
Sum squared resid	2.012680	Schwarz criterion	-5.027082
Log likelihood	10332.12	Hannan-Quinn criter.	-5.034052
Durbin-Watson stat	2.022314		

Component GARCH Model

CTGARCH(1,1) model assuming standardized *t*-Student distribution

Dependent Variable: DLOG(MSFT)
Method: ML - ARCH (Marquardt) - Student's t distribution
Date: 05/10/13 Time: 19:32
Sample (adjusted): 2 4100
Included observations: 4099 after adjustments
Convergence achieved after 18 iterations
Presample variance: backcast (parameter = 0.7)
Q = C(2) + C(3)*(Q(-1) - C(2)) + C(4)*(RESID(-1)^2 - GARCH(-1))
GARCH = Q + (C(5) + C(6)*(RESID(-1)<0))*(RESID(-1)^2 - Q(-1)) + C(7)
*(GARCH(-1) - Q(-1))

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000407	5.59E-05	7.280978	0.0000
Variance Equation				
C(2)	0.047902	0.652276	0.073438	0.9415
C(3)	0.999995	6.96E-05	14369.51	0.0000
C(4)	0.028488	0.005788	4.921913	0.0000
C(5)	0.022151	0.015910	1.392248	0.1638
C(6)	0.091135	0.025313	3.600344	0.0003
C(7)	0.826885	0.036922	22.39536	0.0000
T-DIST. DOF	6.696177	0.465716	14.37823	0.0000

R-squared	-0.000347	Mean dependent var	0.000820
Adjusted R-squared	-0.000347	S.D. dependent var	0.022159
S.E. of regression	0.022162	Akaike info criterion	-5.040913
Sum squared resid	2.012818	Schwarz criterion	-5.028582
Log likelihood	10339.35	Hannan-Quinn criter.	-5.036548
Durbin-Watson stat	2.022176		

Exercises

1. Consider the following outputs obtained with daily log returns of Portuguese government bonds of 10 to 15 years (PT10_P) from January 30, 2007 to January 27, 2012.

Dependent Variable: DLOG(PT10_P)
Method: ML - ARCH (Marquardt) - Student's t distribution
Date: 05/31/13 Time: 14:51
Sample (adjusted): 1/30/2007 1/27/2012
Included observations: 1304 after adjustments
Convergence achieved after 22 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*RESID(-1)^2*(RESID(-1)<0)
+ C(5)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	-0.000300	0.000141	-2.124283	0.0336
Variance Equation				
C	1.18E-06	3.01E-07	3.925820	0.0001
RESID(-1)^2	-0.010924	0.007391	-1.478084	0.1394
RESID(-1)^2*(RESID(-1)<0)	0.124647	0.023881	5.219537	0.0000
GARCH(-1)	0.927791	0.011565	80.22092	0.0000
T-DIST. DOF	4.213684	0.422167	9.981076	0.0000

R-squared	-0.001316	Mean dependent var	-0.000640
Adjusted R-squared	-0.001316	S.D. dependent var	0.009377
S.E. of regression	0.009383	Akaike info criterion	-7.247859
Sum squared resid	0.114729	Schwarz criterion	-7.224056
Log likelihood	4731.604	Hannan-Quinn criter.	-7.238930
Durbin-Watson stat	1.779030		

Exercises

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

Correlogram of the standardized residuals

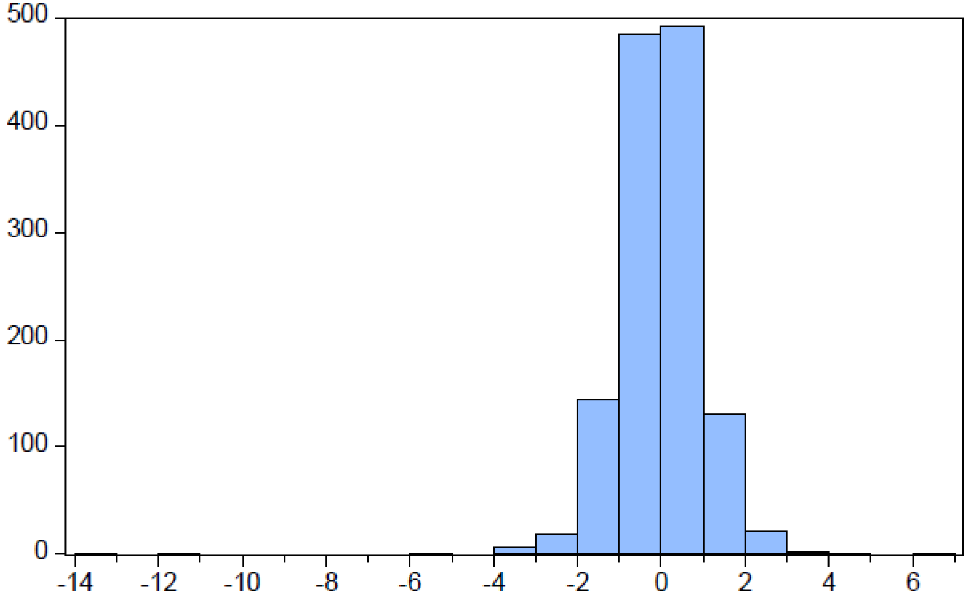
	AC	PAC	Q-Stat	Prob
1	0.135	0.135	23.839	0.000
2	0.003	-0.016	23.850	0.000
3	-0.031	-0.029	25.074	0.000
4	-0.025	-0.018	25.920	0.000
5	-0.032	-0.027	27.249	0.000
6	-0.003	0.004	27.259	0.000
7	0.008	0.006	27.342	0.000
8	0.012	0.008	27.526	0.001
9	0.000	-0.004	27.526	0.001
10	-0.016	-0.016	27.844	0.002
11	0.022	0.028	28.477	0.003
12	-0.008	-0.015	28.568	0.005
13	-0.013	-0.010	28.793	0.007
14	0.018	0.022	29.231	0.010
15	0.023	0.017	29.951	0.012
16	0.025	0.021	30.806	0.014
17	0.009	0.003	30.914	0.020
18	-0.034	-0.036	32.488	0.019
19	-0.008	0.004	32.581	0.027
20	0.008	0.011	32.664	0.037

Correlogram of the squared standardized residuals

	AC	PAC	Q-Stat	Prob
1	-0.001	-0.001	0.0013	0.971
2	-0.006	-0.006	0.0558	0.972
3	0.002	0.002	0.0626	0.996
4	-0.007	-0.007	0.1285	0.998
5	-0.005	-0.005	0.1557	1.000
6	-0.006	-0.007	0.2094	1.000
7	-0.007	-0.007	0.2676	1.000
8	-0.009	-0.010	0.3837	1.000
9	0.006	0.006	0.4280	1.000
10	-0.000	-0.001	0.4282	1.000
11	-0.008	-0.008	0.5200	1.000
12	-0.006	-0.006	0.5641	1.000
13	-0.005	-0.005	0.5939	1.000
14	-0.004	-0.004	0.6181	1.000
15	-0.005	-0.005	0.6542	1.000
16	0.004	0.004	0.6806	1.000
17	-0.000	-0.001	0.6808	1.000
18	-0.009	-0.009	0.7770	1.000
19	0.002	0.001	0.7811	1.000
20	0.002	0.002	0.7876	1.000

Exercises

Histogram of the standardized residuals



Series: Standardized Residuals
Sample 1/30/2007 1/27/2012
Observations 1304

Mean	-0.038737
Median	-0.012123
Maximum	6.507788
Minimum	-13.60066
Std. Dev.	1.068005
Skewness	-2.398487
Kurtosis	33.64390
Jarque-Bera	52271.92
Probability	0.000000

a) Write down and interpret the fitted model.

b) Is there any evidence of serial correlation in the log returns? Is there any evidence of ARCH effects in the log returns?

c) Test the null of whether the standardized residuals are normally distributed.

Exercises

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

2. Consider the following outputs obtained with daily log returns of Spanish government bonds of 1 to 3 years (ES1_3P) from January 30, 2007 to January 27, 2012.

Dependent Variable: DLOG(ES1_3P)
Method: ML - ARCH (Marquardt) - Student's t distribution
Date: 06/27/13 Time: 18:14
Sample (adjusted): 1/30/2007 1/27/2012
Included observations: 1304 after adjustments
Convergence achieved after 17 iterations
Presample variance: backcast (parameter = 0.7)
LOG(GARCH) = C(3) + C(4)*ABS(RESID(-1)/@SQRT(GARCH(-1))) + C(5)*RESID(-1)/@SQRT(GARCH(-1)) + C(6)*LOG(GARCH(-1)) + C(7)*LOG(GARCH(-2))

Variable	Coefficient	Std. Error	z-Statistic	Prob.
@SQRT(GARCH)	0.116110	0.057311	2.025963	0.0428
C	-0.000153	5.41E-05	-2.834956	0.0046
Variance Equation				
C(3)	-0.597101	0.167289	-3.569287	0.0004
C(4)	0.303228	0.071326	4.251302	0.0000
C(5)	-0.098368	0.032094	-3.064976	0.0022
C(6)	0.891768	0.241880	3.686819	0.0002
C(7)	0.079847	0.236722	0.337305	0.7359
T-DIST. DOF	4.505011	0.606979	7.422027	0.0000

R-squared	0.005205	Mean dependent var	6.18E-06
Adjusted R-squared	0.004441	S.D. dependent var	0.001664
S.E. of regression	0.001661	Akaike info criterion	-10.74108
Sum squared resid	0.003590	Schwarz criterion	-10.70934
Log likelihood	7011.182	Hannan-Quinn criter.	-10.72917
Durbin-Watson stat	1.492658		

Exercises

Correlogram of the standardized residuals

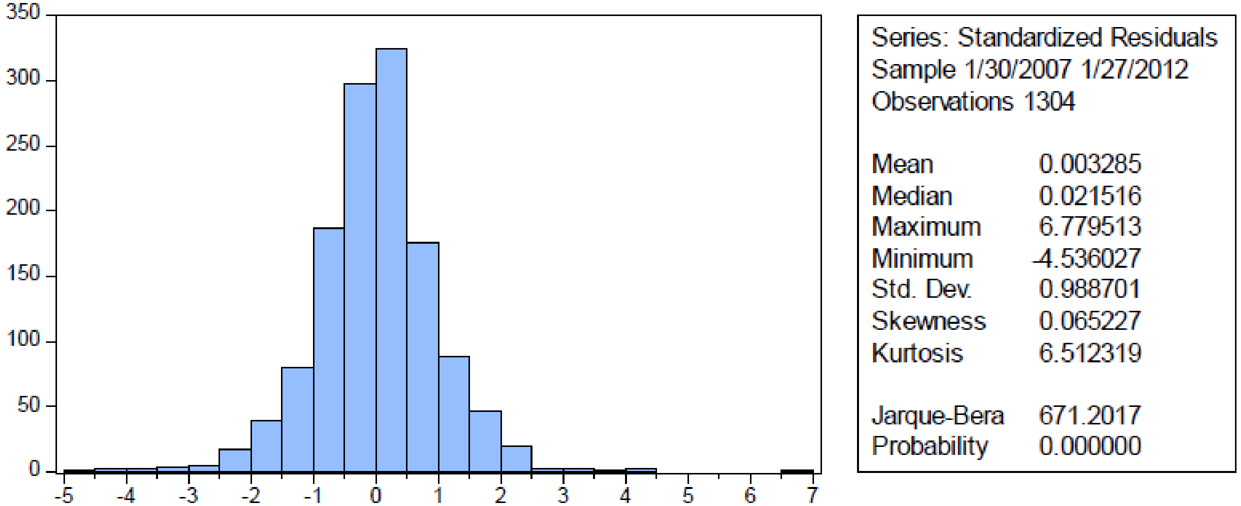
	AC	PAC	Q-Stat	Prob
1	0.174	0.174	39.466	0.000
2	0.008	-0.023	39.550	0.000
3	-0.014	-0.012	39.805	0.000
4	-0.032	-0.029	41.183	0.000
5	-0.003	0.008	41.193	0.000
6	0.002	0.001	41.199	0.000
7	0.007	0.006	41.267	0.000
8	0.031	0.029	42.541	0.000
9	0.021	0.011	43.103	0.000
10	0.054	0.051	46.932	0.000
11	0.013	-0.004	47.163	0.000
12	0.013	0.015	47.400	0.000
13	0.025	0.023	48.214	0.000
14	0.024	0.019	48.957	0.000
15	-0.014	-0.022	49.227	0.000

Correlogram of the squared standardized residuals

	AC	PAC	Q-Stat	Prob
1	-0.005	-0.005	0.0272	0.869
2	0.004	0.004	0.0437	0.978
3	-0.008	-0.008	0.1231	0.989
4	0.026	0.025	0.9793	0.913
5	-0.003	-0.003	0.9905	0.963
6	-0.028	-0.028	1.9937	0.920
7	-0.037	-0.037	3.8233	0.800
8	-0.001	-0.002	3.8245	0.873
9	-0.022	-0.023	4.4868	0.877
10	0.039	0.039	6.4484	0.776
11	-0.005	-0.003	6.4803	0.839
12	0.083	0.082	15.650	0.208
13	-0.027	-0.027	16.594	0.219
14	-0.001	-0.005	16.595	0.278
15	0.023	0.023	17.311	0.301

Exercises

Histogram of the standardized residuals



a) Write down and interpret the fitted model.

b) Is there any evidence of serial correlation in the log returns? Is there any evidence of ARCH effects in the log returns?

c) Test the null of whether the standardized residuals are normally distributed.

Exercises

LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

3. Show that a GARCH(1,2) model is equivalent to an ARCH(∞) model.

4. Show that: $u_t \sim GARCH(2,1) \Rightarrow u_t^2 \sim ARMA(2,2)$.

5. Consider a GARCH(1,1) model:

$$u_t = \sigma_t \varepsilon_t, \quad \text{with } \varepsilon_t \text{ IID } N(0,1)$$
$$\sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + \alpha_1 \varepsilon_{t-1}^2$$

a) Show that ε_t^2 follows an ARMA(1,1) model.

b) What happens when $\alpha_1 + \beta_1 = 1$?